Maclaurin

1. A train leaves K for L at 09:30 while another train leaves L for K at 10:00. The first train arrives in L 40 minutes after the trains pass each other. The second train arrives in K 1 hour and 40 minutes after the trains pass.

Each train travels at a constant speed.

At what time did the trains pass each other?

Solution

Suppose that the first train covers a distance u in one minute and the second train a distance v in one minute, and that the trains pass T minutes after 09:30. Call the crossing point X.

The first train takes T minutes to go from K to X, and the second train takes 100 minutes to travel from X to K. Hence, equating distances, $Tu = 100v$.

The first train takes 40 minutes to travel from X to L, and the second train takes $T - 30$ minutes to travel from L to X, since it started 30 minutes later than the first train. Hence, again equating distances, $40u = (T - 30)v$.

Each of these two equations allows us to find $\frac{u}{v}$. Equating the resulting expressions, we obtain a quadratic equation for T.

We have $\frac{u}{v} = \frac{T - 30}{40} = \frac{100}{T}$.

Rearranging, we get $T^2 - 30T - 4000 = 0$, and factorising, we obtain $(T - 80)(T + 50) = 0$ so that $T = 80$ (ignoring the negative answer, which is not practicable here).

Hence the trains pass at 10:50.

2. A right-angled triangle has area $150\,\text{cm}^2$ and the length of its perimeter is $60\,\text{cm}$.

What are the lengths of its sides?

Solution

Let the sides of the triangle adjacent to the right angle be a cm and b cm. Since the area is $150\,\text{cm}^2$ we have $300 = ab$.

The length of the hypotenuse, in cm, is $60 - (a + b)$, so that, using Pythagoras' theorem, we have

$$a^2 + b^2 = 3600 - 120(a + b) + (a + b)^2.$$

After expanding and simplifying, we get

$$120(a + b) = 3600 + 2ab$$

and so, substituting $300 = ab$, we obtain $a + b = 35$. It follows that a and b are roots of the equation $x^2 - 35x + 300 = 0$, which we may factorise as $(x - 20)(x - 15) = 0$. Therefore a and b are 15 and 20, in either order.

Hence the sides of the triangle have lengths 15 cm, 20 cm and 25 cm.

3. Two numbers are such that the sum of their reciprocals is equal to 1. Each of these numbers is then reduced by 1 to give two new numbers.

Prove that these two new numbers are reciprocals of each other.

[*The* reciprocal *of a non-zero number x is the number* $\frac{1}{x}$.]

Solution

Let the original numbers be x and y. We have

$$\frac{1}{x} + \frac{1}{y} = 1. \tag{*}$$

Multiplying both sides by xy, we obtain

$$y + x = xy.$$

Rearranging, we have

$$1 = xy - x - y + 1,$$

so that

$$1 = (x - 1)(y - 1).$$

Since their product is 1 neither $x - 1$ nor $y - 1$ is 0. Therefore

$$\frac{1}{y-1} = x - 1 \quad \text{and} \quad \frac{1}{x-1} = y - 1,$$

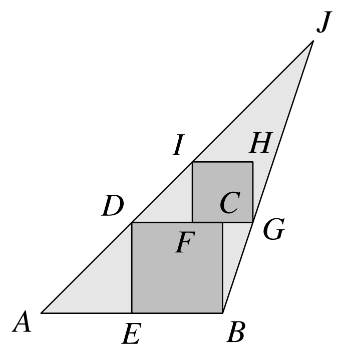
and so the numbers reduced by 1 are reciprocals of each other.

Note

It is essential that the argument here is presented in the order shown, going from what we are given to what we have to prove.

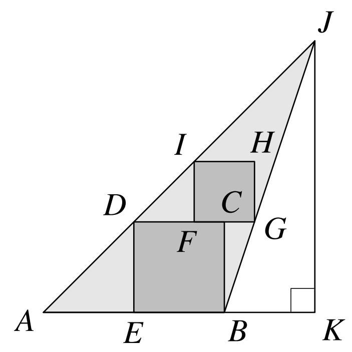
4. The diagram shows the two squares $BCDE$ and $FGHI$ inside the triangle ABJ, where E is the midpoint of AB and C is the midpoint of FG.

What is the ratio of the area of the square $BCDE$ to the area of the triangle ABJ?



Solution

Let K be the foot of the perpendicular from J to AB produced, as shown alongside.



Since E is the midpoint of AB, we have $AE = EB = ED$ and so $\angle DAE = \angle JAK = 45°$.

Let $FG = 2y$ and $BK = x$. Then $FC = CG = y$, $DF = FI = 2y$ and $DC = BC = 3y$.

Triangles GCB and BKJ are similar, since their angles are equal.

Hence $\dfrac{JK}{KB} = \dfrac{CB}{CG} = 3$ and so $JK = 3x$.

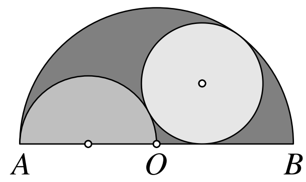
But $JK = AK = 6y + x$. It follows that $3x = 6y + x$ and so $x = 3y$.

The area of $BCDE$ is $9y^2$ and that of ABJ is $27y^2$, so the required ratio is $1 : 3$.

5. A semicircle of radius 1 is drawn inside a semicircle of radius 2, as shown in the diagram, where $OA = OB = 2$.

A circle is drawn so that it touches each of the semicircles and their common diameter, as shown.

What is the radius of the circle?



Solution

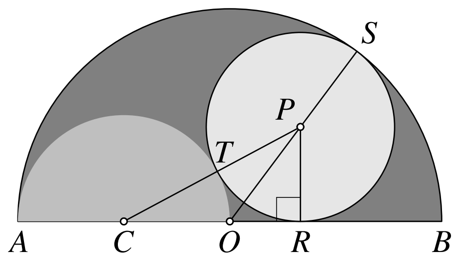
Commentary

A tangent to a circle is perpendicular to the radius through the point of contact; in particular, it follows that the line of centres of two tangent circles passes through the point of tangency. We make use of these facts several times in what follows; for example, when we mention collinearity.

Let the centre of the small semicircle be C, that of the circle be P, and let the circle touch the small semicircle at T, noting that C, T and P are collinear.

Let the circle touch the large semicircle at S. Then O, P and S are also collinear.

Finally let the circle touch the diameter of the large semicircle at R, so that $\angle PRC = 90°$, as shown.



We have $OA = 2$ and $OC = 1$. Let the radius of the circle be r, so that $PO = 2 - r$, and let $OR = x$.

Now, applying Pythagoras' theorem to triangle ORP, we get $x^2 + r^2 = (2 - r)^2$. It follows that

$$x^2 = 4 - 4r. \tag{1}$$

Also, applying Pythagoras' theorem to triangle CRP, we have $(1 + x)^2 + r^2 = (1 + r)^2$. It follows that

$$2x + x^2 = 2r. \tag{2}$$

Hence, from equations (1) and (2), $x^2 = 4 - 2(2x + x^2)$ and so $3x^2 + 4x - 4 = 0$, that is, $(3x - 2)(x + 2) = 0$. Taking the positive root (the negative root being impracticable here), we have $x = \frac{2}{3}$ and so $r = \frac{8}{9}$.

Thus the radius of the circle is $\frac{8}{9}$.

6. A tiling of an $n \times n$ square grid is formed using 4×1 tiles.

What are the possible values of n?

[A tiling *has no gaps or overlaps, and no tile goes outside the region being tiled.*]

SOLUTION

Each tile is of area 4, so the area of the grid is a multiple of 4. Hence n is even. We consider two cases, determined by whether n is a multiple of 4 or not.

$n = 4k$, for some positive integer k

In this case the grid can be tiled using $4k^2$ tiles, with each row containing k tiles. Figure 1 shows the case $k = 3$.

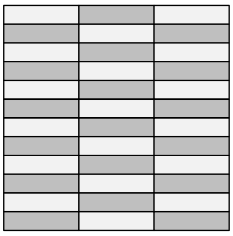


Figure 1

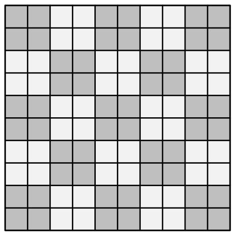


Figure 2

$n = 4k - 2$, for some positive integer k

In this case the grid cannot be tiled. To see this, colour the grid in a 2×2 chessboard pattern, as shown in Figure 2, where $k = 3$.

Notice that, in whatever way a 4×1 tile is placed on the coloured grid, it will cover two cells of each colour. So if it were possible to tile the grid using 4×1 tiles, then they would cover an equal number of 1×1 squares of each colour.

However, there is an odd number of 2×2 squares in the grid, so that the number of 1×1 squares of one colour in the coloured grid is always four more than the number of 1×1 squares of the other colour.

Hence such a tiling is impossible.

Therefore the values of n for which it is possible to tile an $n \times n$ grid with 4×1 tiles are the multiples of 4 (only).